Quotient Groups
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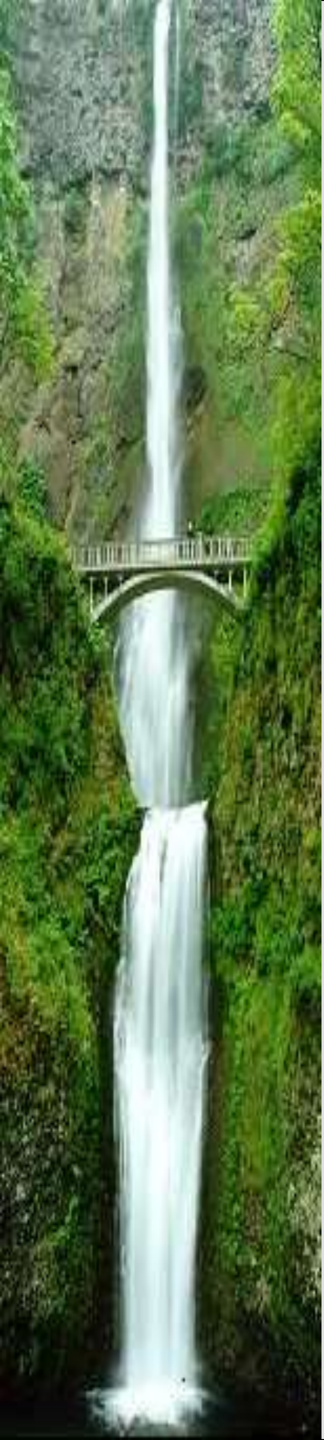

Evens and Odds

- We know that (Z, +) is a group.
- We previously observed that this group could be broken up into two pieces to form a two element group:

	EVENS {...-6,-4,-2,0,2,4,6,...)	ODDS {...-5,-3 -1,1,3,5,7,...)
EVENS {...-6,-4,-2,0,2,4,6,...)	EVENS {...-6,-4,-2,0,2,4,6,...)	ODDS {...-5,-3 -1,1,3,5,7,...)
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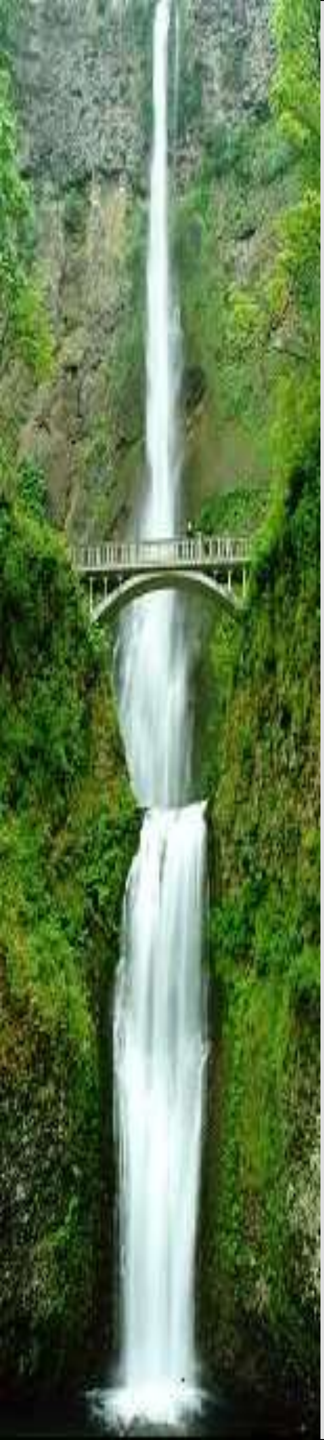


This is a group!

- The elements are subsets of the original group.
- You add two subsets together like so:

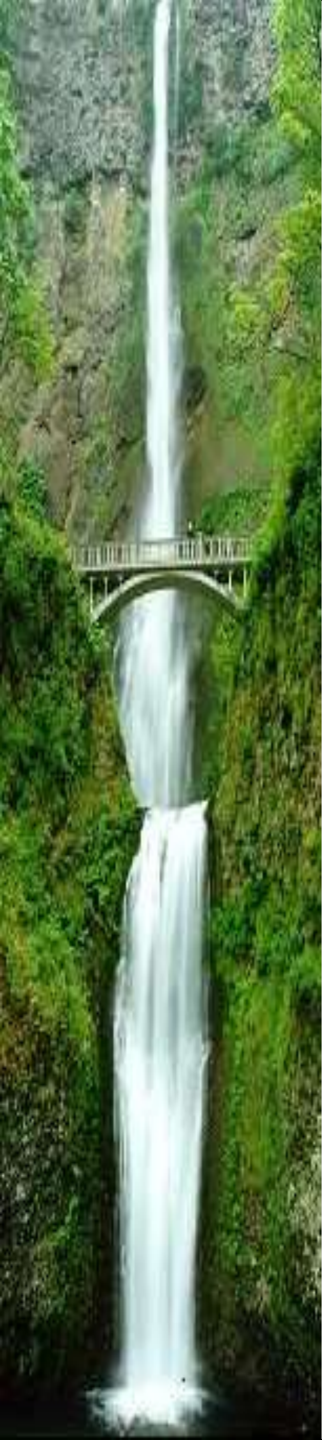
$$\text{EVENS} + \text{ODDS} = \{n + m \mid n \in \text{EVENS} \ \& \ m \in \text{ODDS}\}$$

(We found a more efficient way to add these subsets later!)

Flips and Rotations

- We figured out that we could do the same thing with D_8, the group of symmetries of a square:

	Rotations $\{I, R, R^2, R^3\}$	Flips $\{F, FR, FR^2, FR^3\}$
Rotations $\{I, R, R^2, R^3\}$	Rotations $\{I, R, R^2, R^3\}$	Flips $\{F, FR, FR^2, FR^3\}$
Flips $\{F, FR, FR^2, FR^3\}$	Flips $\{F, FR, FR^2, FR^3\}$	Rotations $\{I, R, R^2, R^3\}$



This is a group!

- The elements are subsets of D8
- You multiply two subsets together like so:

$$\text{Flips} \times \text{Rotations} = \{ab \mid a \in \text{Flips} \ \& \ b \in \text{Rotations}\}$$

- In general, given two subsets, A and B, of a group G we can define their product as:

$$AB = \{ab \mid a \in A \ \& \ b \in B\}$$

(for cosets, though, there's an easier way)

We found, in D8, two other ways to make a 2-element group this way:

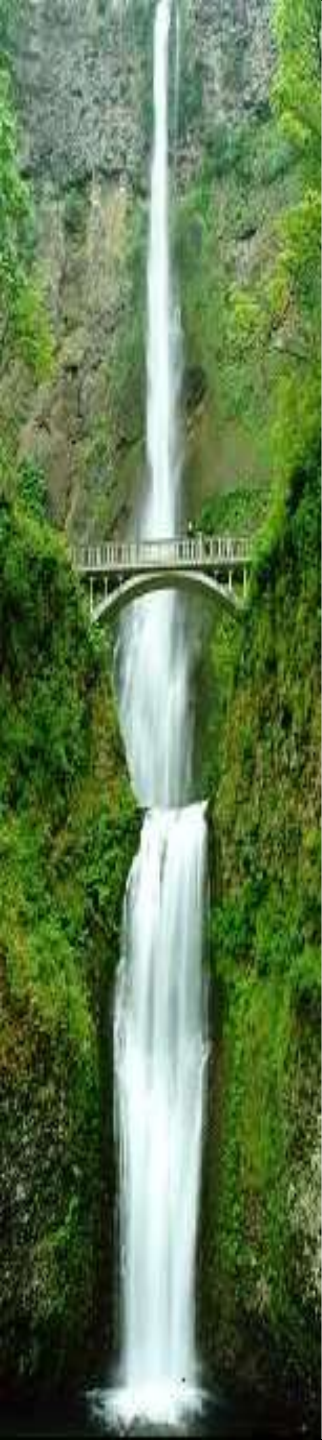
- We could break it into:

$\{I, R^2, F, FR^2\}$ and $\{R, R^3, FR, FR^3\}$

- Or we could break it into:

$\{I, R^2, FR, FR^3\}$ and $\{R, R^3, F, FR^2\}$

People noted that the "EVENs" needed to be a subgroup (or at least closed so that EVEN EVEN = EVEN).



Breaking D8 into smaller pieces to make a group with 4 elements

We only found one way that would work

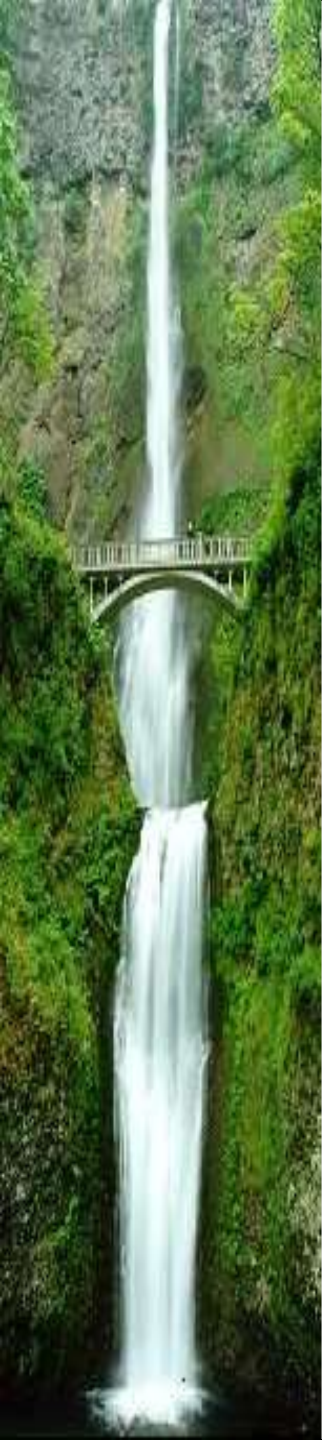
	$\{I, R^2\}$	$\{R, R^3\}$	$\{F, FR^2\}$	$\{FR, FR^3\}$
$\{I, R^2\}$	$\{I, R^2\}$	$\{R, R^3\}$	$\{F, FR^2\}$	$\{FR, FR^3\}$
$\{R, R^3\}$	$\{R, R^3\}$	$\{I, R^2\}$	$\{FR, FR^3\}$	$\{F, FR^2\}$
$\{F, FR^2\}$	$\{F, FR^2\}$	$\{FR, FR^3\}$	$\{I, R^2\}$	$\{R, R^3\}$
$\{FR, FR^3\}$	$\{FR, FR^3\}$	$\{F, FR^2\}$	$\{R, R^3\}$	$\{I, R^2\}$

These groups of subsets are called *quotient groups*

- What is needed to be able to construct a quotient group?
- We found out that:
  - One of the subsets has to be a subgroup
  - All of the subsets have to be the same size
  - The subsets can not overlap
  - All group elements must be used

(Left) Cosets

- We determined how to find all of the subsets given the subgroup:
  - Multiply a group element by the subgroup to get one of your subsets.
- For example, using the subgroup $H=\{I, R^2\}$ of D8, we would:
  - Multiply by F (or FR^2)to get the subset F $H= \{F, FR^2\}$
  - Multiply by R (or R^3) to get the subset R $H=\{R, R^3\}$
  - Multiply by FR (or FR^3) to get FR $H= \{FR, FR^3\}$
- These subsets are called *cosets* of H in D8.



Coset notation

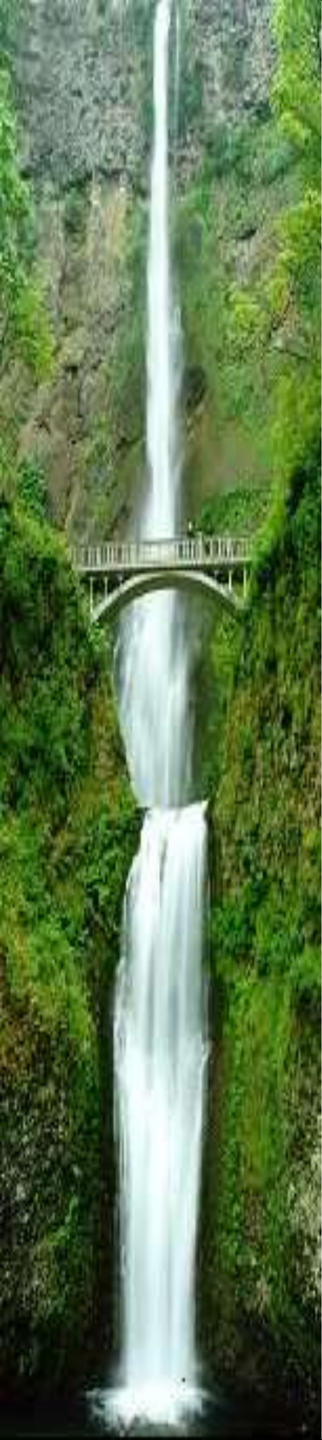
- Let G be a group and H a subgroup of G.

- The set of left cosets of H in G is denoted

$$G/H = \{gH \mid g \in \mathrm{G}\}$$

where $gH = \{gh \mid h \in H\}$

- We also use this for the quotient group, when the cosets of H actually form one.

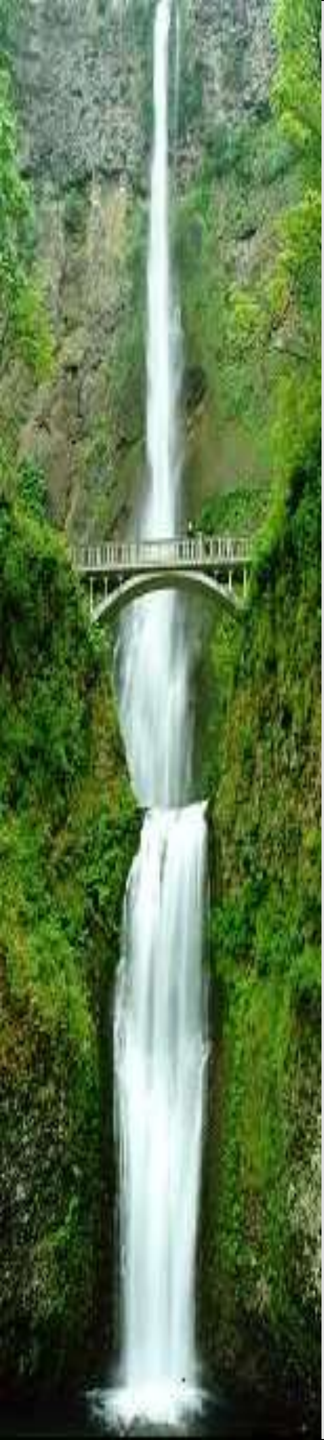


Normal Subgroups

- We found out that not all subgroups can be used to make quotient groups.
  - For example, {I, F} cannot be used to make a quotient group in D_8.
- We figured out that, in order for *H* to work, we needed $gH=Hg$ for each element of *g*.
- Subgroups H that satisfy this condition are called normal subgroups:

Definition: Let *G* be a group and *H* a subgroup of *G*. Then *H* is a *normal* subgroup of *G* if

$$gH=Hg \quad \forall g\in G.$$



Why is $gH = Hg$ for all $g \in G$ needed?

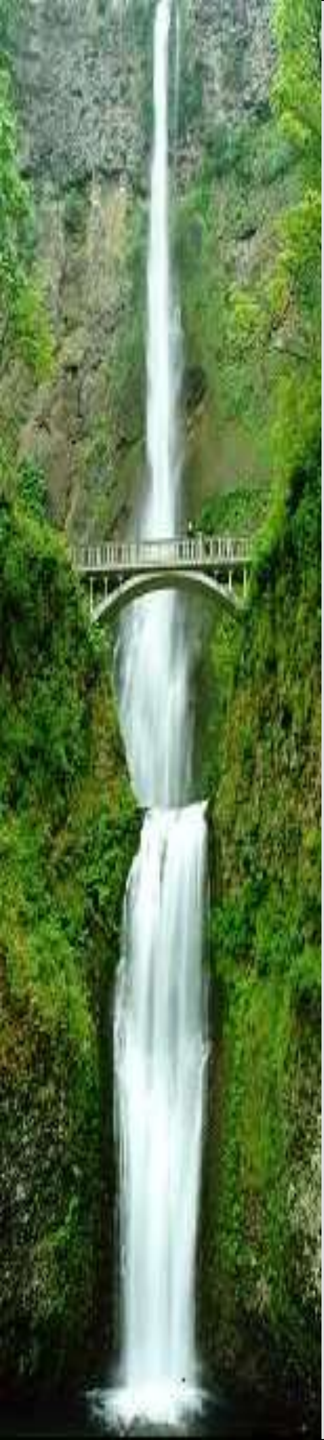
Consider the case of $H = \{I,F\}$ in D_8:

	$\{I,F\}$	$\{R,FR^3\}$	$\{R^2,FR^2\}$	$\{R^3,FR\}$
$\{I,F\}$	$\{I,F\}$	Uh OH!	Uh OH!	Uh OH!
$\{R,FR^3\}$	$\{R,FR^3\}$			
$\{R^2,FR^2\}$	$\{R^2,FR^2\}$			
$\{R^3,FR\}$	$\{R^3,FR\}$			

When we get to the second spot in row one, we have to multiply by R on the *right* of H. This gives us different stuff than what is in $RH = \{R, FR^3\}$ which was created by multiplying by R on the *left* of H.

But H is supposed to be the identity!

(We should get $RH\,H = H\,RH = RH$)



An easier way to multiply cosets:

Definition: Let G be a group and H a normal subgroup of G. If $a, b \in G$, then:

$$(aH)\,(bH) = abH.$$

Does this match our original way of thinking about multiplying cosets?

….yes, ***if*** H is normal!

Theorem: Let G be a group and H a subgroup. Then G/H is a group (under the operation of coset multiplication) if and only if H is normal.

Proof: see last time!